A New Formalization of Power Series in Coq

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The Coquelicot project

- Goal :
  - build a user-friendly library of real analysis in Coq.

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  - build a user-friendly library of real analysis in Coq.
- Previous work [CPP' 2012] :
  - total functions to easily write limits, derivatives and integrals,
  - tactic to automatize proofs of differentiability.

A few words about limits of sequences

Definition of limit in the style of the standard library:

Definition Lim_seq $(u_n)_{n\in\mathbb{N}}$
(pr : {l : R | Un_cv$(u_n)_{n\in\mathbb{N}}$l}) :=
projT1 pr.

with dependent type

A few words about limits of sequences

Definition `Lim_seq` $(u_n)_{n \in \mathbb{N}}$ `:=`

$$\frac{\overline{\lim}\,(u_n) + \underline{\lim}\,(u_n)}{2} \in \overline{\mathbb{R}}$$

total function

without dependent type

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total function

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Some other user-friendly definitions:

- $\underset{t\to x}{\texttt{Lim}}\; f(t) :=$ `Lim_seq` $(f(x_n))_{n\in\mathbb{N}} \in \overline{\mathbb{R}}$ when $\lim (x_n)_{n\in\mathbb{N}} = x$
- `Derive` f `(x : R)` := $\underset{h\to 0}{\text{Lim}} \left(\frac{f(x+h) - f(x)}{h}\right) \in \mathbb{R}$
- `RInt` f `(a b : R)` := `Lim_seq` $\left(\frac{b-a}{n}\sum_{k=0}^{n} f(x_k)\right)_{n\in\mathbb{N}} \in \mathbb{R}$

Some applications

- D'Alembert Formula [CPP' 2012]
  $$u(x,t) = \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)\,d\xi + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\,d\tau$$
  $$\frac{\partial^2 u}{\partial t^2}(x,t) - c\frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t)$$
- Convergence of a sequence based on algebraic-geometric means [Bertot 2013]
  $a_0 = 1,\ b_0 = \frac{1}{x},\ a_{n+1} = \frac{a_n+b_n}{2},\ b_{n+1} = \sqrt{a_n b_n}$ and
  $f(x) = \lim a_n = \lim b_n \Rightarrow \pi = 2\sqrt{2}\, f\left(\frac{1}{\sqrt{2}}\right) \Big/ f'\left(\frac{1}{\sqrt{2}}\right)$
- Baccalaureate of Mathematics 2013 [BAC 2013]
  $$\int_{\frac{1}{e}}^{1} \frac{2+2\ln x}{x}\,dx = 1$$

Motivations to build power series

Some of the many uses of power series:

- basic functions (e^x, sin, cos, ...),
- solutions for differential equations,
- equivalent functions,
- generating functions, ...

$\Rightarrow$ must be formalized in a library of real analysis.

Coq standard library

- about sequences
  - two different definitions for limits toward finite limit and $+\infty$
  - limits of sums, opposites, products, and multiplicative inverses of sequences in the finite case
- about power series
  - series of real numbers provide convergence criteria
  - sequences of functions provide continuity and differentiability

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  - arithmetic operations on power series
  - integrability of power series

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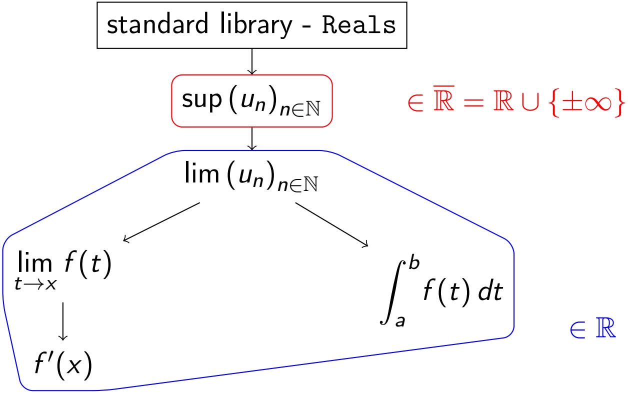
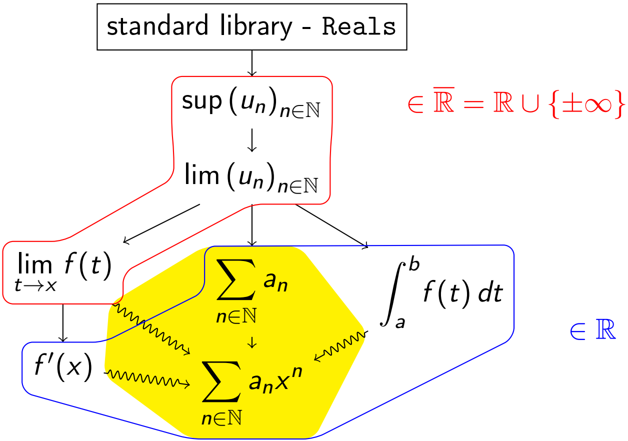
Coquelicot library – CPP version



Coquelicot library – present version



Definition

- Series:

$$\texttt{Series}\ (a_n)_{n\in\mathbb{N}} = \texttt{Lim\_seq}\ \left(\sum_{k=0}^{n} a_k\right)_{n\in\mathbb{N}}$$

- Power series:

$$\texttt{PSeries}\ (a_n)_{n\in\mathbb{N}} = \texttt{Series}\ \left(a_k x^k\right)_{n\in\mathbb{N}}$$

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inherit all the good properties of `Lim_seq`

- easy to write
- some rewritings without hypothesis

Use-case: Bessel Functions

$$J_n = \left(\frac{x}{2}\right)^n \sum_{p=0}^{+\infty} \frac{(-1)^p}{p!(n+p)!} \left(\left(\frac{x}{2}\right)^2\right)^p$$

Use-case: Bessel Functions

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- $J_n''(x) + x \cdot J_n'(x) + (x^2 - n^2) \cdot J_n(x) = 0$
- $J_{n+1}(x) = \dfrac{n \cdot J_n(x)}{x} - J_n'(x)$
- $J_{n+1}(x) - J_{n-1}(x) = \dfrac{2n}{x} J_n(x)$
- $J_{n+1}(x) - J_{n-1}(x) = -2 \cdot J_n'(x)$

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Example: differential equation

$$J_n''(x) + x \cdot J_n'(x) + (x^2 - n^2)J_n(x) = 0$$

Needed operations on power series:

Example: differential equation

with $a_p^{(n)} = \dfrac{(-1)^p}{p!(n+p)!}$ and $X = \left(\dfrac{x}{2}\right)^2$:

$$\left(\left(\frac{x}{2}\right)^n \sum_{p=0}^{+\infty} a_p^{(n)} X^p\right)'' + x \cdot \left(\left(\frac{x}{2}\right)^n \sum_{p=0}^{+\infty} a_p^{(n)} X^p\right)' + (x^2 - n^2) \left(\frac{x}{2}\right)^n \sum_{p=0}^{+\infty} a_p^{(n)} X^p = 0$$

Needed operations on power series:

- function to write power series

Example: differential equation

with $a_p^{(n)} = \dfrac{(-1)^p}{p!(n+p)!}$ and $X = \left(\dfrac{x}{2}\right)^2$:

$$X\left(\sum_{p=0}^{+\infty} a_p^{(n)} X^p\right)'' + (n+1)\left(\sum_{p=0}^{+\infty} a_p^{(n)} X^p\right)' + \sum_{p=0}^{+\infty} a_p^{(n)} X^p = 0$$

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with $a_p^{(n)} = \dfrac{(-1)^p}{p!(n+p)!}$ and $X = \left(\dfrac{x}{2}\right)^2$:

$$X \sum_{p=0}^{+\infty} \left((p+1)(p+2)a_{p+2}^{(n)} X^p\right) + (n+1)\sum_{p=0}^{+\infty} \left((p+1)a_{p+1}^{(n)} X^p\right) + \sum_{p=0}^{+\infty} a_p^{(n)} X^p = 0$$

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- function to write power series
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Example: differential equation

with $a_p^{(n)} = \dfrac{(-1)^p}{p!(n+p)!}$ and $X = \left(\dfrac{x}{2}\right)^2$:

$$\sum_{p=0}^{+\infty} \left(p(p+1)a_{p+1}^{(n)} X^p\right) + (n+1)\sum_{p=0}^{+\infty} \left((p+1)a_{p+1}^{(n)} X^p\right) + \sum_{p=0}^{+\infty} a_p^{(n)} X^p = 0$$

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- function to write power series
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Example: differential equation

with $a_p^{(n)} = \dfrac{(-1)^p}{p!(n+p)!}$ and $X = \left(\dfrac{x}{2}\right)^2$:

$$\forall p \in \mathbb{N}, \quad p(p+1)a_{p+1}^{(n)} + (n+1)(p+1)a_{p+1}^{(n)} + a_p^{(n)} = 0$$

Needed operations on power series:

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- variable multiplication
- arithmetic operations
- extensionality

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Unicity

$$X\left(\sum_{p=0}^{+\infty} a_p^{(n)} X^p\right)'' + (n+1)\left(\sum_{p=0}^{+\infty} a_p^{(n)} X^p\right)' + \sum_{p=0}^{+\infty} a_p^{(n)} X^p = 0$$

$$\Uparrow$$

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$$\Updownarrow$$

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Operations on Series

- scalar multiplication:
  $c \cdot \sum_{n\in\mathbb{N}} a_n = \sum_{n\in\mathbb{N}} (c \cdot a_n)$, without hypothesis.
- index shift: $\forall k \in \mathbb{N}^*,\quad \sum_{n=0}^{k-1} a_n + \sum_{n\in\mathbb{N}} a_{n+k} = \sum_{n\in\mathbb{N}} a_n,$
  if $\sum a_n$ are convergent or $\forall n < k, a_n = 0$.

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- addition: $\sum_{n\in\mathbb{N}} a_n + \sum_{n\in\mathbb{N}} b_n = \sum_{n\in\mathbb{N}} (a_n + b_n)$,
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- multiplication: $\sum_{n\in\mathbb{N}} a_n \cdot \sum_{n\in\mathbb{N}} b_n = \sum_{n\in\mathbb{N}} \left(\sum_{k=0}^{n} a_k \cdot b_{n-k} \right)$,
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Operations on Power Series

- scalar multiplication:
$$c \cdot \sum_{n\in\mathbb{N}} a_n x^n = \sum_{n\in\mathbb{N}} (c \cdot a_n)\, x^n$$
- multiplication by a variable:
$$\forall k \in \mathbb{N}, \quad x^k \cdot \sum_{n\in\mathbb{N}} a_n x^n = \sum_{n\in\mathbb{N}} a_{n-k} x^n$$

} **No hypothesis**

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  if $\sum |a_n x^n|$ and $\sum |b_n x^n|$ are convergent.

Some features related to power series

- Convergence circle
- Differentiability
- Sequences of functions

Convergence circle

$$\mathcal{C}_a = \sup \left\{ r \in \mathbb{R} \;\middle|\; \sum |a_n r^n| \text{ is convergent} \right\} \in \overline{\mathbb{R}}$$

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Formally proved:

- Equality with $\sup \{ r \in \mathbb{R} \mid |a_n r^n| \text{ is bounded} \}$
- Compatibility with operations (*e.g.*: $\mathcal{C}_{a+b} \geq \min \{\mathcal{C}_a, \mathcal{C}_b\}$)

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- Compatibility with operations (*e.g.*: $\mathcal{C}_{a+b} \geq \min \{\mathcal{C}_a, \mathcal{C}_b\}$)
- If $|x| < \mathcal{C}_a$, then $\sum a_n x^n$ is absolutely convergent
- If $|x| > \mathcal{C}_a$, then $\sum a_n x^n$ is strongly divergent

Differentiability

To write

$$\text{If } |x| < \mathcal{C}_a, \text{ then } \left(\sum_{n\in\mathbb{N}} a_n x^n\right)' = \sum_{n\in\mathbb{N}} (n+1)a_{n+1}x^n:$$

using the Coq standard library:

```
Lemma Derive_PSeries (a : nat -> R) (cv_a : R) :
  forall (PS : forall x : R, Rabs x < cv_a -> {l : R | Pser a x l})
    (PS' : forall x : R, Rabs x < cv_a ->
           {l : R | Pser (fun n : nat => INR (S n) * a (S n)) x l})
    (pr : forall x : R, Rabs x < cv_a ->
        derivable_pt (fun y : R =>
           match Rlt_dec (Rabs y) cv_a with
           | left Hy => projT1 (PS y Hy)
           | right _ => 0
           end) x)
    (x : R) (Hx : Rabs x < cv_a),
  derive_pt (fun y : R =>
    match Rlt_dec (Rabs y) cv_a with
    | left Hy => projT1 (PS y Hy)
    | right _ => 0
    end) x (pr x Hx) = projT1 (PS' x Hx)
```

Differentiability

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using the Coquelicot library:

```
Lemma Derive_PSeries (a : nat -> R) :
  forall x : R, Rbar_lt (Rabs x) (CV_circle a) ->
    Derive (PSeries a) x
      = PSeries (fun n : nat => INR (S n) * a (S n)) x.
```

Differentiability

To write

$$\text{If } |x| < \mathcal{C}_a \text{, then } \left(\sum_{n\in\mathbb{N}} a_n x^n\right)^{(k)} = \sum_{n\in\mathbb{N}} \frac{(n+k)!}{n!} a_{n+k} x^n:$$

using the Coquelicot library:

```
Lemma Derive_n_PSeries (k : nat) (a : nat -> R) :
  forall x : R, Rbar_lt (Rabs x) (CV_circle a) ->
    Derive_n (PSeries a) n x
      = PSeries (fun n : nat =>
           (INR (fact (n + k)) / INR (fact n)) * a (n + k)) x.
```

Sequences of function

Useful for:

- Power series $\sum a_n x^n$
- Fourier series $\sum a_n \cos(nx) + b_n \sin(nx)$
- ...

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Formally proved:

- limits:

$$\forall \left(f_n\right)_{n\in\mathbb{N}} \text{ a sequence of functions, } D \text{ an open subset of } \mathbb{R},$$
$$\text{if } \left(f_n\right)_{n\in\mathbb{N}} \text{ is uniformly convergent and}$$
$$\forall x \in D, \forall n \in \mathbb{N}, \lim_{t\to x} f(t) \text{ exists, then}$$
$$\forall x \in D, \lim_{t\to x}\left(\lim_{n\to+\infty}(f_n(t))\right) = \lim_{n\to+\infty}\left(\lim_{t\to x} f_n(t)\right)$$

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- continuity
- differentiability

Power series in other proof assistants

C-CoRN, HOL Light, Isabelle/HOL, PVS:

- two different notions of finite and infinite convergence circle
- series of real numbers provide
  - various convergence theorems
- sequences of functions provide
  - differentiability
  - integrability

Power series in other proof assistants

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but results are not explicitly power series

Conclusion

New power series for Coq:

- easy to use:

$$\begin{cases} \mathcal{C}_{J_n} = +\infty & : \text{ 41 LoC} \\ J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x) & : \text{ 35 LoC} \\ x^2 \cdot J_n''(x) + x \cdot J_n'(x) + (x^2 - n^2) \cdot J_n(x) = 0 & : \text{ 94 LoC} \end{cases}$$

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- with a proper notion of convergence circle

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	Nb. Definitions	Nb. Lemmas	Nb. Lines
Series	3	47	764
PSeries	13	70	1674

Available at : http://coquelicot.saclay.inria.fr/

Perspectives

- About Power Series:
  - Composition
  - Quotient
  - Automation

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  - Composition
  - Quotient
  - Automation
- About Real Analysis:
  - Left and right limits
  - Equivalent functions
  - Automation for limits, integrals and equivalents
- To go further: complex numbers

Build a user-friendly library of real analysis in Coq.

Any questions?

http://coquelicot.saclay.inria.fr/

Bibliography

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- Catherine Lelay
  www.lri.fr/~lelay/

Limits' troubles

Definition `Lim_seq` $(u_n)_{n\in\mathbb{N}}$:= $\dfrac{\overline{\lim}\,(u_n) + \underline{\lim}\,(u_n)}{2} \in \overline{\mathbb{R}}$

- `Lim_seq` $(-1)^n = 0$
- $\underset{x\to 0}{\texttt{Lim\_fct}}\ x^{-1} = +\infty$

As on paper: can be written,
but no meaning without proof of convergence

Left and right limits

Actual alternative on left and right limits:

$$\lim_{x\to 0^+} x^{-1} = \underset{x\to 0}{\texttt{Lim\_fct}}\ |x|^{-1} \text{ and } \lim_{x\to 0^+} x^{-1} = \underset{x\to 0}{\texttt{Lim\_fct}}\ (-|x|)^{-1}$$